

ON EQUIVARIANT PRINCIPAL BUNDLES OVER WONDERFUL COMPACTIFICATIONS

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ABSTRACT. Let G be a simple algebraic group of adjoint type over \mathbb{C} , and let M be the wonderful compactification of a symmetric space G/H . Take a \tilde{G} -equivariant principal R -bundle E on M , where R is a complex reductive algebraic group and \tilde{G} is the universal cover of G . If the action of the isotropy group \tilde{H} on the fiber of E at the identity coset is irreducible, then we prove that E is polystable with respect to any polarization on M . Further, for wonderful compactification of the quotient of $\mathrm{PSL}(n, \mathbb{C})$, $n \neq 4$ (respectively, $\mathrm{PSL}(2n, \mathbb{C})$) by the normalizer of the projective orthogonal group (respectively, the projective symplectic group), we prove that the tangent bundle is stable with respect to any polarization on the wonderful compactification.

1. INTRODUCTION

Let G be a semi-simple linear algebraic group of adjoint type defined over the field \mathbb{C} of complex numbers. The universal cover of G will be denoted by \tilde{G} . Let σ be an algebraic involution of G induced by an automorphism $\tilde{\sigma}$ of \tilde{G} of order two. The fixed point subgroup of G for σ will be denoted by H . The quotient G/H is an affine variety. De Concini and Procesi constructed a compactification of G/H which is known as the wonderful compactification [DP]. The left-translation action of G on G/H extends to an action of G on the wonderful compactification $\overline{G/H}$. This produces an action of \tilde{G} on $\overline{G/H}$. Our aim here is to investigate the \tilde{G} -equivariant principal bundles on $\overline{G/H}$.

Let R be a connected reductive complex linear algebraic group. Let $E \rightarrow \overline{G/H}$ be a \tilde{G} -equivariant algebraic principal R -bundle. The inverse image of H in \tilde{G} will be denoted by \tilde{H} . Since the isotropy for the point $e = eH \in G/H$ for the action of \tilde{G} is \tilde{H} , we have an action of \tilde{H} on the fiber E_e . Let

$$(1.1) \quad \gamma : \tilde{H} \longrightarrow \mathrm{Aut}^R(E_e)$$

be the corresponding homomorphism, where $\mathrm{Aut}^R(E_e)$ is the group of algebraic automorphisms of E_e that commute with the action of R on it. The groups $\mathrm{Aut}^R(E_e)$ and R are isomorphic by an isomorphism which is unique up to inner automorphisms.

We prove the following (see Proposition 2.1):

If $\gamma(\tilde{H})$ is not contained in any proper parabolic subgroup of $\mathrm{Aut}^R(E_e)$, then the principal R -bundle E is polystable with respect to every polarization of $\overline{G/H}$.

2000 *Mathematics Subject Classification.* 32Q26, 14M27, 14M17.

Key words and phrases. Wonderful compactification, equivariant principal bundle, tangent bundle, stability.

For equivariant vector bundles on $\overline{G/H}$, the above proposition can be improved; see Proposition 3.3 for the precise statement.

In Section 4, we consider the following two symmetric spaces $\mathrm{PSL}(n, \mathbb{C})/\mathrm{NPSO}(n, \mathbb{C})$, $n \neq 4$ and $\mathrm{PSL}(2m, \mathbb{C})/\mathrm{NPSp}(2m, \mathbb{C})$, $m \geq 2$, where $\mathrm{NPSO}(n, \mathbb{C})$ (respectively, $\mathrm{PSp}(n, \mathbb{C})$) denote the normalizer of the projective orthogonal group (respectively, projective symplectic group) in $\mathrm{PSL}(n, \mathbb{C})$ (respectively, $\mathrm{PSL}(2m, \mathbb{C})$). See [DP, p. 7, Lemma], for details.

The first one of the above two symmetric spaces corresponds to the involution σ of $\mathrm{PSL}(n, \mathbb{C})$ induced by the automorphism

$$A \mapsto (A^t)^{-1}$$

of $\mathrm{SL}(n, \mathbb{C})$. The second one corresponds to the involution σ of $\mathrm{PSL}(2m, \mathbb{C})$ induced by the automorphism

$$A \mapsto J^{-1}(A^t)^{-1}J$$

of $\mathrm{SL}(2m, \mathbb{C})$, where

$$J := \begin{pmatrix} 0 & I_{m \times m} \\ -I_{m \times m} & 0 \end{pmatrix}.$$

In Theorem 4.1, and Remark 4.2 we prove the following:

For the wonderful compactification $\overline{G/H}$ of the above two symmetric spaces, the tangent bundle $T\overline{G/H}$ is stable with respect to every polarization of $\overline{G/H}$.

As pointed out by the referee, Theorem 4.1 remains valid as long as the three conditions stated in the beginning of Section 3 are valid and the H -module $\mathrm{Lie}(G)/\mathrm{Lie}(H)$ is irreducible (see Remark 4.3).

2. POLYSTABILITY OF IRREDUCIBLE EQUIVARIANT BUNDLES

We continue with the above notation. The wonderful compactification $\overline{G/H}$ of the quotient G/H will be denoted by M . The left-translation action of G on G/H extends to an action $G \times M \rightarrow M$. Using the natural projection $\tilde{G} \rightarrow G$ from the universal cover, the above action of G on M produces an action

$$(2.1) \quad \rho : \tilde{G} \times M \rightarrow M$$

of \tilde{G} on M .

Let R be a connected reductive complex linear algebraic group. An equivariant principal R -bundle on M is an algebraic principal R -bundle on M equipped with a lift of the left-action of \tilde{G} in (2.1). More precisely, an equivariant R -bundle is a pair $(E, \tilde{\rho})$, where $E \rightarrow M$ is an algebraic principal R -bundle, and

$$\tilde{\rho} : \tilde{G} \times E \rightarrow E$$

is an algebraic action of \tilde{G} on the total space of E , such that the following two conditions hold:

- (1) the projection of E to M intertwines the actions of \tilde{G} on E and M , and
- (2) the action of \tilde{G} on E commutes with the action of R on E .

Let $(E, \tilde{\rho})$ be an equivariant principal R -bundle on M . Let

$$\mathrm{Ad}(E) := E \times^R R \longrightarrow M$$

be the fiber bundle associated to E for the conjugation action of R on itself. Since the conjugation action of R on itself preserves the group structure of R , the fibers of $\mathrm{Ad}(E)$ are groups isomorphic to R . To see an explicit isomorphism of R with a fiber $\mathrm{Ad}(E)_x$, where $x \in X$, fix a point $z_0 \in E_x$. Now the map

$$(2.2) \quad R \longrightarrow \mathrm{Ad}(E)_x$$

that sends any $g \in R$ to the equivalence class of $(z_0, g) \in E \times R$ is an isomorphism of groups. Therefore, $\mathrm{Ad}(E)_x$ is identified with R by an isomorphism which is unique up to an inner automorphism of R .

The equivariant R -bundle $(E, \tilde{\rho})$ is called *irreducible* if the image $\gamma(\tilde{H})$ is not contained in any proper parabolic subgroup of $\mathrm{Aut}^R(E_e) = \mathrm{Ad}(E)_e$, where $\tilde{H} \subset \tilde{G}$, as before, is the inverse image of H , and γ is the homomorphism in (1.1).

Fix a polarization $\mathcal{L} \in H^2(M, \mathbb{Q})$ on M , meaning \mathcal{L} is the class of a very ample line bundle on M . The degree of a torsionfree coherent sheaf on M is defined using \mathcal{L} as follows: for a torsionfree coherent sheaf F on M ,

$$\mathrm{degree}(F) := (c_1(F) \cdot \mathcal{L}^{n-1}) \cap [M] \in \mathbb{Z},$$

where n is the (complex) dimension of M .

An algebraic vector bundle V on M is called *semistable* (respectively, *stable*) if for every coherent subsheaf $F \subset V$ with $0 < \mathrm{rank}(F) < \mathrm{rank}(V)$, the inequality

$$\mu(F) := \frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} \leq \mu(V) := \frac{\mathrm{degree}(V)}{\mathrm{rank}(V)}$$

(respectively, $\mu(F) < \mu(V)$) holds. A semistable vector bundle is called *polystable* if it is a direct sum of stable vector bundles.

A principal R -bundle E on M is called *semistable* (respectively, *stable*) if for every maximal proper parabolic subgroup $P \subset R$ and for every reduction $\tau : U \longrightarrow (E|_U)/P$ over some Zariski open set U of M such that the codimension of $M \setminus U$ is at least two, we have $\mathrm{degree}(\tau^* T_{\mathrm{rel}}) \geq 0$ (respectively, $\mathrm{degree}(\tau^* T_{\mathrm{rel}}) > 0$), where T_{rel} is the relative tangent bundle for the natural projection of $(E/P)|_U$ to U . A principal R -bundle E on M is said to be *polystable* if there is a parabolic subgroup P of R and a reduction of structure group $E_L \subset E$ to a Levi factor $L \subset P$ such that

- (1) the principal L bundle E_L on M is stable, and
- (2) the principal P -bundle obtained by extending the structure group of E_L

$$E_P := E_L \times^L P$$

has the property that for any character χ of P which is trivial on the center of R , the line bundle on M associated to E_P for χ has degree zero.

Proposition 2.1. *Let $(E, \tilde{\rho})$ be an irreducible equivariant R -bundle on M . Then the principal R -bundle E is polystable.*

Proof. We will first prove that E is semistable. Assume that E is not semistable. Then, there is a Zariski open subset $U \subset M$ such that the complement $M \setminus U$ is of codimension at least two, a proper parabolic subgroup $P \subset R$, and an algebraic reduction of structure group

$$E_P \subset E|_U$$

of E to P over U , such that E_P is the Harder-Narasimhan reduction for E (see [AAB] for Harder-Narasimhan reduction). Let

$$\mathrm{Ad}(E_P) := E_P \times^P P \longrightarrow U$$

be the adjoint bundle associated to E_P for the conjugation action of P on itself. Just as for $\mathrm{Ad}(E)$, the fibers of $\mathrm{Ad}(E_P)$ are groups isomorphic to P because the conjugation action of P on itself preserves the group structure. The natural inclusion of $E_P \times P$ in $(E|_U) \times G$ produces an inclusion

$$\mathrm{Ad}(E_P) \hookrightarrow \mathrm{Ad}(E)|_U.$$

In the isomorphism in (2.2) if we take $z_0 \in (E_P)_x$, then the isomorphism sends P isomorphically to the fiber $\mathrm{Ad}(E_P)_x$. Therefore, $\mathrm{Ad}(E_P)_x$ is a parabolic subgroup of $\mathrm{Ad}(E)_x$ because P is a parabolic subgroup of R .

Take an element $g \in \tilde{G}$ such that the point

$$\bar{g} := \rho(g, e) \in G/H \subset M$$

lies in the above open subset $U \subset M$. Consider the automorphism of E defined by $z \mapsto \tilde{\rho}(g, z)$. The subgroup $g\tilde{H}g^{-1} \subset \tilde{G}$ preserves the fiber $E_{\bar{g}}$ because $g\tilde{H}g^{-1}$ is the isotropy of \bar{g} for the action ρ in (2.1). Therefore, we get a homomorphism

$$(2.3) \quad \theta : g\tilde{H}g^{-1} \longrightarrow \mathrm{Ad}(E)_{\bar{g}}.$$

The action of g on E produces an isomorphism of algebraic groups

$$(2.4) \quad \eta : \mathrm{Ad}(E)_e \longrightarrow \mathrm{Ad}(E)_{\bar{g}}.$$

The following diagram is commutative

$$(2.5) \quad \begin{array}{ccc} \tilde{H} & \xrightarrow{\gamma} & \mathrm{Ad}(E)_e \\ \downarrow g' & & \downarrow \eta \\ g\tilde{H}g^{-1} & \xrightarrow{\theta} & \mathrm{Ad}(E)_{\bar{g}} \end{array}$$

where γ , θ and η are defined in (1.1), (2.3) and (2.4) respectively, and $g'(y) = gyg^{-1}$.

The polarization \mathcal{L} on M is preserved by the action of \tilde{G} in (2.1) because \tilde{G} is connected. The action of \tilde{G} on E preserves the pair U and E_P because the Harder-Narasimhan reduction is unique for a given polarization. Consequently, the image $\theta(g\tilde{H}g^{-1})$ in (2.3) is contained in the parabolic subgroup

$$(2.6) \quad \mathrm{Ad}(E_P)_{\bar{g}} \subset \mathrm{Ad}(E)_{\bar{g}}.$$

On the other hand, since $\gamma(\tilde{H})$ is not contained in any proper parabolic subgroup of $\mathrm{Ad}(E)_e$, from the commutativity of the diagram in (2.5) we conclude that $\theta(g\tilde{H}g^{-1})$ is

not contained in any proper parabolic subgroup of $\mathrm{Ad}(E)_{\bar{G}}$. But this is in contradiction with (2.6). Therefore, we conclude that the principal R -bundle E is semistable.

We will now prove that E is polystable. Note that E is polystable if and only if the adjoint vector bundle $\mathrm{ad}(E) := E \times^R \mathfrak{R}$ is polystable, where \mathfrak{R} is the Lie algebra of the reductive group R (see [AB, p. 224, Corollary 3.8]). Thus it is enough to prove that $\mathrm{ad}(E)$ is polystable. To prove that $\mathrm{ad}(E)$ is polystable, we simply replace the Harder–Narasimhan reduction in the above proof by the socle reduction of the semistable vector bundle $\mathrm{ad}(E)$ (see [AB, p. 218, Proposition 2.12]). Since the socle reduction is unique, simply repeating the above proof we get that $\mathrm{ad}(E)$ is polystable. \square

3. EQUIVARIANT VECTOR BUNDLES

Let G and H be as before. From now on we will assume the following:

- (1) The connected component $\tilde{H}^0 \subset \tilde{H}$ containing the identity element is a simple algebraic group.
- (2) For any maximal torus T_0 of the connected component $\tilde{H}^0 \subset \tilde{H}$ containing the identity element, for any Borel subgroup B_0 of \tilde{H}^0 containing T_0 , for any Borel subgroup B of \tilde{G} containing B_0 and for any maximal torus T of B containing T_0 , the restriction map $X(B)^+ \rightarrow X(B_0)^+$ is surjective. Here $X(B)$ (respectively, $X(B)^+$) denotes the set of all characters (respectively, dominant characters) of B . Similarly $X(B_0)^+$ is defined.
- (3) The restriction of any simple root α of \tilde{G} (with respect to T and B) to B_0 is nonzero and it is a nonnegative integral linear combination of simple roots of \tilde{H}^0 (with respect to T_0 and B_0).

Remark 3.1. The pairs of groups $G = \mathrm{PSL}(n, \mathbb{C})$, $H = \mathrm{NPSO}(n, \mathbb{C})$, $n \neq 2, 4$ and $G = \mathrm{PSL}(2m, \mathbb{C})$, $H = \mathrm{PSp}(2m, \mathbb{C})$, $m \geq 2$, satisfy the above conditions (1), (2), (3). To see this we consider $\mathrm{SO}(n, \mathbb{C})$ (respectively, $\mathrm{Sp}(2m, \mathbb{C})$) as the subgroup of the special linear group preserving the nondegenerate symmetric (respectively, skew-symmetric) bilinear form

$$\sum_{i=1}^n X_i Y_{n+1-i} \text{ (respectively, } \sum_{i=1}^m (X_i Y_{2m+1-i} - X_{m+i} Y_{m+1-i}) \text{)}.$$

- (1) The simplicity of the above groups H follows from the facts about the classical groups of type A, B, C and D .
- (2) By choice of the nondegenerate bilinear forms we see that the inclusion of maximal torus $T_0 \subset T$ and the inclusion of Borel subgroup $B_0 \subset B$ satisfies the hypothesis (2), (3) above (see [FH, p. 215, p. 243, p. 272]).

Lemma 3.2. *Every irreducible representation of \tilde{H}^0 is a restriction of some representation of \tilde{H} .*

Proof. Let V be an irreducible representation of \tilde{H}^0 . If V is the trivial representation, then there is nothing to prove. Otherwise, let λ be the highest weight of V . Then, by using the part (2) of the hypothesis, there is a dominant character χ of B whose restriction

to B_0 is λ . Hence, the irreducible representation $V(\chi)$ of \tilde{G} with highest weight χ is a direct sum of V with multiplicity one and of some irreducible representations of \tilde{H}^0 with highest weights μ satisfying $\mu < \lambda$ for the dominant ordering in \tilde{H}^0 . This is because every weight ν of $V(\chi)$ satisfies $\nu < \chi$ for the dominant ordering in \tilde{G} with respect to T and B and by using the part (3) of the hypothesis that the restriction to B_0 of every simple root α of \tilde{G} with respect to T and B is nonzero and is a nonnegative integral linear combination of simple roots of \tilde{H}^0 .

Since any two Borel subgroups of \tilde{H}^0 are conjugate in \tilde{H}^0 and any two maximal tori of B_0 are conjugate in B_0 , we may choose the representatives of \tilde{H}/\tilde{H}^0 to lie in both $N_{\tilde{H}}(B_0)$ and $N_{\tilde{H}}(T_0)$. Consequently, the finite group \tilde{H}/\tilde{H}^0 acts on the group of characters of B_0 preserving the dominant characters (not necessarily preserving pointwise). Further, since the representatives of \tilde{H}/\tilde{H}^0 can be chosen in $N_{\tilde{H}}(B_0)$ the action of \tilde{H}/\tilde{H}^0 preserves the positive roots of \tilde{H}^0 with respect to B_0 . Thus, the \tilde{H} -span of V in $V(\chi)$ is a direct sum of irreducible representations of \tilde{H}^0 whose highest weights are of the form $\sigma(\lambda)$ with σ running over the elements of the finite group \tilde{H}/\tilde{H}^0 . By the previous paragraph, for every $\sigma \in \tilde{H}/\tilde{H}^0$, either $\sigma(\lambda) < \lambda$ or $\sigma(\lambda) = \lambda$. On the other hand, if for some $\sigma \in \tilde{H}/\tilde{H}^0$ we have $\sigma(\lambda) < \lambda$ then

$$\lambda = \sigma^n(\lambda) < \sigma^{n-1}(\lambda) < \dots < \sigma(\lambda) < \lambda,$$

where n is the order of σ , which is a contradiction. Therefore, the \tilde{H} -span of V coincides with V , implying that V is a restriction of a representation of \tilde{H} . \square

A vector bundle W of rank r on M is called *equivariant* if W corresponds to an equivariant principal $\mathrm{GL}(r, \mathbb{C})$ -bundle. Equivalently, an equivariant vector bundle is a pair $(W, \tilde{\rho})$, where W is an algebraic vector bundle on M , and

$$\tilde{\rho} : \tilde{G} \times W \longrightarrow W$$

is an algebraic action of \tilde{G} on W , such that the following two conditions hold:

- (1) the projection of W to M intertwines the actions of \tilde{G} on W and M , and
- (2) the action $\tilde{\rho}$ preserves the linear structure of the fibers of W .

An equivariant vector bundle $(W, \tilde{\rho})$ is called *irreducible* if the representation

$$\rho_e : \tilde{H} \longrightarrow \mathrm{GL}(W_e)$$

given by the action of the isotropy subgroup for the point $e \in M$ is irreducible. Note that the irreducible equivariant vector bundles of rank r correspond to the irreducible equivariant principal $\mathrm{GL}(r, \mathbb{C})$ -bundles.

Proposition 3.3. *Let $(W, \tilde{\rho})$ be an irreducible equivariant vector bundle on M of rank r . Then either W is stable, or W admits a decomposition*

$$W = L^{\oplus r},$$

where L is a line bundle on M .

Proof. The vector bundle W is polystable by Proposition 2.1. Therefore, W can be uniquely decomposed as

$$(3.1) \quad W = \bigoplus_{i=1}^{\ell} W_i \otimes_{\mathbb{C}} H^0(M, W \otimes W_i^{\vee}),$$

where W_i are distinct stable vector bundles on M . The above assertion of uniqueness means the following: if

$$W = \bigoplus_{j=1}^{\ell'} W'_j \otimes_{\mathbb{C}} \mathbb{C}^{r_j},$$

where $W'_1, \dots, W'_{\ell'}$ are non-isomorphic stable vector bundles, then $\ell = \ell'$ and there is a permutation α of $\{1, \dots, \ell\}$ such that the subbundle $W_i \otimes_{\mathbb{C}} H^0(M, W \otimes W_i^{\vee})$ of W in (3.1) coincides with the above subbundle

$$W'_{\alpha(i)} \otimes_{\mathbb{C}} \mathbb{C}^{r_{\alpha(i)}} \subset W.$$

In particular, W_i is isomorphic to $W'_{\alpha(i)}$ and $\dim H^0(M, W \otimes W_i^{\vee}) = r_{\alpha(i)}$. This uniqueness follows immediately from the Krull–Schmidt decomposition of a vector bundle (see [At, p. 315, Theorem 3]) and the fact that for any two non-isomorphic stable vector bundles W^1 and W^2 ,

$$H^0(M, W^1 \otimes (W^2)^{\vee}) = 0.$$

For any $g \in \tilde{G}$, let

$$(3.2) \quad \rho_g : M \longrightarrow M$$

be the automorphism defined by $x \mapsto \rho(g^{-1}, x)$. Similarly, let

$$(3.3) \quad \tilde{\rho}_g : W \longrightarrow W$$

be the map defined by $v \mapsto \tilde{\rho}(g, v)$; note that $\tilde{\rho}_g$ is an isomorphism of the vector bundle W with the pullback $\rho_g^* W$. The pullback

$$\rho_g^* W = \bigoplus_{i=1}^{\ell} \rho_g^*(W_i \otimes_{\mathbb{C}} H^0(M, W \otimes W_i^{\vee}))$$

of the decomposition in (3.1) coincides with the unique decomposition (unique in the above sense) of $\rho_g^* W$. Hence the isomorphism $\tilde{\rho}_g$ takes the above decomposition of $\rho_g^* W$ to a permutation $\nu(g)$ of the decomposition of W in (3.1). Therefore, we get a map

$$\nu : \tilde{G} \longrightarrow P(\ell),$$

where $P(\ell)$ is the group of permutations of $\{1, \dots, \ell\}$, that sends any $g \in \tilde{G}$ to the above permutation $\nu(g)$. This map ν is clearly continuous, the permutation $\nu(e)$ is the identity map of $\{1, \dots, \ell\}$, and \tilde{G} is connected. These together imply that ν is the constant map to the identity map of $\{1, \dots, \ell\}$. In other words, the action of \tilde{G} on W preserves the subbundle

$$W_i \otimes_{\mathbb{C}} H^0(M, W \otimes W_i^{\vee}) \subset W$$

in (3.1) for every i .

We will next show that each vector bundle W_i admits a \tilde{G} equivariant structure.

We have noted above that the action of \tilde{G} on W preserves the subbundle $W_i \otimes_{\mathbb{C}} H^0(M, W \otimes W_i^\vee)$. The automorphism $\tilde{\rho}_g$ of $W_i \otimes_{\mathbb{C}} H^0(M, W \otimes W_i^\vee)$ (see (3.3)) produces an isomorphism

$$(\rho_g^* W_i) \otimes_{\mathbb{C}} H^0(M, W \otimes W_i^\vee) \xrightarrow{\sim} W_i \otimes_{\mathbb{C}} H^0(M, W \otimes W_i^\vee),$$

where ρ_g is defined in (3.2). Since the vector bundle W_i is indecomposable, the vector bundle $\rho_g^* W_i$ is also indecomposable, and hence from [At, p. 315, Theorem 2] we know that W_i is isomorphic to $\rho_g^* W_i$.

Take any integer $1 \leq i \leq \ell$. Let $\text{Aut}(W_i)$ denote the group of all algebraic automorphisms of the vector bundle W_i . It should be clarified that any automorphism of W_i lying in $\text{Aut}(W_i)$ is over the identity map of M . The group $\text{Aut}(W_i)$ is the Zariski open subset of the affine space $H^0(M, W_i \otimes W_i^\vee)$ defined by the locus of invertible endomorphisms. Therefore, $\text{Aut}(W_i)$ is a connected complex algebraic group.

Let $\widetilde{\text{Aut}}(W_i)$ denote the set of all pairs of the form (g, f) , where $g \in \tilde{G}$ and

$$f : \rho_g^* W_i \longrightarrow W_i$$

is an algebraic isomorphism of vector bundles, where ρ_g is the automorphism in (3.2). This set $\widetilde{\text{Aut}}(W_i)$ has a tautological structure of a group

$$(g_2, f_2) \cdot (g_1, f_1) = (g_1 g_2, f_2 \circ \rho_{g_2}^*(f_1)).$$

Since the vector bundle W_i is simple (recall that it is stable), the group $\widetilde{\text{Aut}}(W_i)$ fits in a short exact sequence of algebraic groups

$$(3.4) \quad e \longrightarrow \mathbb{C}^* \longrightarrow \widetilde{\text{Aut}}(W_i) \xrightarrow{\delta_i} \tilde{G} \longrightarrow e,$$

where δ_i sends any (g, f) to g . Note that the earlier observation that W_i is isomorphic to $\rho_g^* W_i$ for all $g \in \tilde{G}$ implies that the homomorphism δ_i in (3.4) is surjective.

The Lie algebras of \tilde{G} and $\widetilde{\text{Aut}}(W_i)$ will be denoted by \mathfrak{g} and $A(W_i)$ respectively. Let

$$(3.5) \quad \delta'_i : A(W_i) \longrightarrow \mathfrak{g}$$

be the homomorphism of Lie algebras corresponding to δ_i in (3.4). Since \mathfrak{g} is semisimple, there is a homomorphism of Lie algebras

$$\tau_i : \mathfrak{g} \longrightarrow A(W_i)$$

such that

$$(3.6) \quad \delta'_i \circ \tau_i = \text{Id}_{\mathfrak{g}}$$

[Bo, p. 91, Corollaire 3]. Fix a homomorphism $\tau_i : \mathfrak{g} \longrightarrow A(W_i)$ satisfying (3.6). Since the group \tilde{G} is simply connected, there is a unique algebraic representation

$$\tilde{\tau}_i : \tilde{G} \longrightarrow \widetilde{\text{Aut}}(W_i)$$

such that the corresponding homomorphism of Lie algebras coincides with τ_i . From (3.6) it follows immediately that $\delta_i \circ \tilde{\tau}_i = \text{Id}_{\tilde{G}}$.

We now note that $\tilde{\tau}_i$ defines an action of \tilde{G} on W_i . The pair $(W_i, \tilde{\tau}_i)$ is an equivariant vector bundle. In particular, the fiber $(W_i)_e$ is a representation of \tilde{H} .

Consider the decomposition of W in (3.1). The actions of \tilde{G} on W and W_i together define a linear action of \tilde{G} on $H^0(M, W \otimes W_i^\vee)$. With respect to these actions, the isomorphism in (3.1) is \tilde{G} -equivariant.

Since the isomorphism in (3.1) is \tilde{G} -equivariant, we get an isomorphism of representations of \tilde{H}

$$W_e = \bigoplus_{i=1}^{\ell} (W_i)_e \otimes_{\mathbb{C}} H^0(M, W \otimes W_i^\vee);$$

we noted above that both $(W_i)_e$ and $H^0(M, W \otimes W_i^\vee)$ are representations of \tilde{H} . Since the \tilde{H} -module W_e is irreducible, we conclude that $\ell = 1$. So

$$W = W_1 \otimes H^0(M, W \otimes W_1^\vee),$$

and we have an isomorphism of representations of \tilde{H}

$$W_e = (W_1)_e \otimes H^0(M, W \otimes W_1^\vee).$$

As in Section 1, let $\tilde{\sigma}$ be the lift of σ to \tilde{G} .

By Lemma 3.2, any irreducible \tilde{H}^0 module V of W_e is a restriction of a \tilde{H} module. Hence, it follows that the irreducible \tilde{H} module W_e is an irreducible \tilde{H}^0 module as well.

Recall the assumption that H^0 is a simple algebraic group. From the irreducibility of the \tilde{H}^0 -module W_e it now follows that

- either $\dim H^0(M, W \otimes W_1^\vee) = 1$, or
- $\text{rank}(W_1) = 1$

(see [BK, p. 1469, Lemma 3.2]).

We now observe that if $\dim H^0(M, W \otimes W_1^\vee) = 1$, then $W = W_1$ is stable. On the other hand, if $\text{rank}(W_1) = 1$, then

$$W = W_1^{\oplus r},$$

where r is the rank of W . This completes the proof of the proposition. \square

4. ORTHOGONAL AND SYMPLECTIC QUOTIENT OF SL_n

In this section we consider the wonderful compactification $\overline{G/H}$ of the following two symmetric spaces corresponding to the orthogonal and symplectic structures:

The first one corresponds to the involution σ of $G = \text{PSL}(n, \mathbb{C})$ induced by the automorphism

$$A \mapsto (A^t)^{-1}$$

of $\text{SL}(n, \mathbb{C})$, $n \neq 2, 4$. The connected component of $H = G^\sigma$ is the projective orthogonal group $\text{PSO}(n, \mathbb{C})$.

The second one corresponds to the involution σ of $\text{PSL}(2m, \mathbb{C})$ induced by the automorphism

$$A \mapsto J^{-1}(A^t)^{-1}J$$

of $\mathrm{SL}(2m, \mathbb{C})$, where

$$(4.1) \quad J := \begin{pmatrix} 0 & I_{m \times m} \\ -I_{m \times m} & 0 \end{pmatrix}.$$

In this case, we have $H := G^\sigma = \mathrm{PSp}(2m, \mathbb{C})$.

Theorem 4.1. *For the above two cases, the tangent bundle of $\overline{G/H}$ is stable with respect to any polarization on $\overline{G/H}$.*

Proof. The Lie algebras of G and H will be denoted by \mathfrak{g} and \mathfrak{h} respectively. Consider the natural action of H on $\mathfrak{g}/\mathfrak{h}$. We will show that the H -module $\mathfrak{g}/\mathfrak{h}$ is irreducible.

First consider the case corresponding to the symplectic structure. In this case, $G = \mathrm{PSL}(2m, \mathbb{C})$ and $H = \mathrm{PSp}(2m, \mathbb{C})$. Let

$$\omega \in \bigwedge^2 \mathbb{C}^{2m}$$

be the standard symplectic form given by the matrix J in (4.1). Using the symplectic form ω , we identify $\mathrm{End}(\mathbb{C}^{2m}) = \mathbb{C}^{2m} \otimes (\mathbb{C}^{2m})^\vee$ with $\mathbb{C}^{2m} \otimes \mathbb{C}^{2m}$. Note that this decomposes as

$$\mathbb{C}^{2m} \otimes \mathbb{C}^{2m} = \mathrm{Sym}^2(\mathbb{C}^{2m}) \oplus \bigwedge^2 \mathbb{C}^{2m}.$$

The $\mathrm{PSp}(2m, \mathbb{C})$ -module $\mathfrak{g}/\mathfrak{h}$ is isomorphic to the $\mathrm{PSp}(2m, \mathbb{C})$ -module $(\bigwedge^2 \mathbb{C}^{2m})/\mathbb{C} \cdot \omega$. It is known that the $\mathrm{PSp}(2m, \mathbb{C})$ -module $(\bigwedge^2 \mathbb{C}^{2m})/\mathbb{C} \cdot \omega$ is irreducible [FH, p. 260, Theorem 17.5] (from [FH, Theorem 17.5] it follows immediately that the $\mathrm{PSp}(2m, \mathbb{C})$ -module $\bigwedge^2 \mathbb{C}^{2m}$ is the direct sum of a trivial $\mathrm{PSp}(2m, \mathbb{C})$ -module of dimension one and an irreducible $\mathrm{PSp}(2m, \mathbb{C})$ -module).

Next consider the case corresponding to the orthogonal structure. So $G = \mathrm{PSL}(n, \mathbb{C})$ and $\mathrm{PO}(n, \mathbb{C})$ is the connected component of H containing the identity element. Using the standard orthogonal form

$$\omega' \in \mathrm{Sym}^2(\mathbb{C}^n)$$

on \mathbb{C}^n , identify $\mathbb{C}^n \otimes (\mathbb{C}^n)^\vee$ with

$$\mathbb{C}^n \otimes \mathbb{C}^n = \mathrm{Sym}^2(\mathbb{C}^n) \oplus \bigwedge^2 \mathbb{C}^n.$$

Now the H -module $\mathfrak{g}/\mathfrak{h}$ is isomorphic to the H -module $\mathrm{Sym}^2(\mathbb{C}^n)/\mathbb{C} \cdot \omega'$. It is known that the H -module $\mathrm{Sym}^2(\mathbb{C}^n)/\mathbb{C} \cdot \omega'$ is irreducible [FH, p. 296, Ex. 19.21] (from [FH, p. 296, Ex. 19.21] it follows that the H -module $\mathrm{Sym}^2(\mathbb{C}^n)$ is the direct sum of a trivial H -module of dimension one and an irreducible H -module).

Fix a polarization on $\overline{G/H}$. Let r be the dimension of $\overline{G/H}$.

The action of G on M gives an action of the isotropy subgroup H on the tangent space $T_e \overline{G/H}$. We note that the H -module $T_e \overline{G/H}$ is isomorphic to the H -module $\mathfrak{g}/\mathfrak{h}$. Since the H -module $\mathfrak{g}/\mathfrak{h}$ is irreducible, from Proposition 3.3 and Remark 3.1 we conclude that either the tangent bundle $T \overline{G/H}$ is stable or $T \overline{G/H}$ is isomorphic to $L^{\oplus r}$ for some line bundle L on $\overline{G/H}$.

Now using an argument in [BK] it can be shown that $T \overline{G/H}$ is not of the form $L^{\oplus r}$. Nevertheless, we reproduce the argument below in order to be self-contained.

Assume that $T\overline{G/H}$ is isomorphic to $L^{\oplus r}$. The variety $\overline{G/H}$ is unirational, because G is so. Hence $\overline{G/H}$ is simply connected [Se, p. 483, Proposition 1]. As $T\overline{G/H}$ holomorphically splits into a direct sum of line bundles and $\overline{G/H}$ is simply connected, it follows that

$$\overline{G/H} = (\mathbb{CP}^1)^r$$

[BPT, p. 242, Theorem 1.2]. But the tangent bundle of $(\mathbb{CP}^1)^r$ is not of the form $L^{\oplus r}$. Therefore, $T\overline{G/H}$ is not of the form $L^{\oplus r}$. This completes the proof. \square

Remark 4.2. The wonderful compactification of $\mathrm{PSL}(2, \mathbb{C})/\mathrm{NPSO}(2, \mathbb{C})$ is isomorphic to \mathbb{P}^2 . The tangent bundle of \mathbb{P}^2 is known to be stable (see [PW]).

We thank the referee for pointing out the following:

Remark 4.3. The proof of Theorem 4.1 remains valid if the three conditions stated in the beginning of Section 3 are valid and the H -module $\mathfrak{g}/\mathfrak{h}$ is irreducible. Therefore, the tangent bundle of $\overline{G/H}$ is stable with respect to any polarization on $\overline{G/H}$ if the H -module $\mathfrak{g}/\mathfrak{h}$ is irreducible. If G/H is a non-Hermitian symmetric space, then the H -module $\mathfrak{g}/\mathfrak{h}$ is irreducible.

Remark 4.4. In Remark 4.3 the hypothesis of simplicity of \tilde{H}^0 is necessary. For example, in the case of wonderful compactification of $\mathrm{PSL}(4, \mathbb{C})/\mathrm{NPSO}(4, \mathbb{C})$ we can not use the arguments at the end of the proof of Proposition 3.3. Though the H -module $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{sl}(2, \mathbb{C}) \otimes \mathfrak{sl}(2, \mathbb{C})$ is irreducible, it is a tensor product of two non-trivial irreducible representations, where $\mathfrak{sl}(2, \mathbb{C})$ is the Lie algebra of $\mathrm{PSL}(2, \mathbb{C})$. For the identification of $\mathrm{PSO}(4, \mathbb{C})$ with $\mathrm{PSL}(2, \mathbb{C}) \times \mathrm{PSL}(2, \mathbb{C})$ (see [FH, p. 369]).

ACKNOWLEDGEMENTS

We are very grateful to the referee for comments to improve the exposition. In particular, Remark 4.3 is due to the referee. The first-named author acknowledges support of the J. C. Bose Fellowship.

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